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Quantum defragmentation algorithm

Daniel Burgarth[1] and Vittorio Giovannetti[2]
[1] *IMS and QOLS, Imperial College, London SW72BK, UK*
[2] *NEST, Scuola Normale Superiore and CNR-INFM, Piazza dei Cavalieri 7, I-56126 Pisa, Italy*

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In this addendum to our paper [D. Burgarth and V. Giovannetti, Phys. Rev. Lett. **99**, 100501 (2007)] we prove that during the transformation that allows one to enforce control by relaxation on a quantum system, the ancillary memory can be kept at a finite size, independently from the fidelity one wants to achieve. The result is obtained by introducing the quantum analog of defragmentation algorithms which are employed for efficiently reorganizing classical information in conventional hard disks.

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In recent years, increasing attention has been devoted to developing schemes that allow one to achieve global control on a large many-body quantum system $V = C \cup \overline{C}$ by only having direct access to a relatively small subpart C [1–11]. The majority of results obtained so far have been derived within the general framework of an "algebraic" approach to control theory where the allowed operations are parametrized by specifying which (local) components of the system Hamiltonian can be manipulated via proper choices of classical pulses (e.g., see Ref. [12]). An independent approach was recently proposed by us in Refs. [7,8], introducing the notion of local controllability of quantum systems via "relaxation." In this scheme a method of controlling $V = C \cup \overline{C}$ by acting on C *and* on an external memory system M was suggested. This is essentially achieved by transferring the states of V into M through a sequence of iterative operational steps (see Fig. 1) which induces an effective relaxation of V into the memory degree of freedom. The states are then controlled in M and, by using the inverted sequence of steps, transferred back to V.

Such a method can be important in inhomogeneous scenarios, where some parts (e.g., M) are easier to control than others (e.g., V). It also allows for an easy-to-check criterion to determine whether a given system V is controllable, which can be applied to large systems [7,8] and which was subsequently generalized to the algebraic control scenario [6]. Finally, compared to algebraic control, the scheme proposed in Refs. [7,8] has the advantage that the control protocol is constructive and follows a clear physical intuition.

The main drawback of the controllability by relaxation approach stems from the fact that it cannot reduce the size of the controlled system (in contrast to algebraic control). Indeed to be able to transfer arbitrary states from V to the ancillary memory M the latter must be at least as large as the former (i.e., $\dim M \geqslant \dim V$). Even more problematic is the fact that up till now no upper bounds were known on the minimal size of M which is needed to accomplish the control. In this paper we fix this problem by showing that M can be kept at a finite size, which is maximally twice as large as V. This is a major improvement to [7,8], where M was arbitrarily large. The result is derived by introducing the quantum analog of defragmentation algorithms. In computer science, defragmentation is a process that allows one to reduce the amount of fragmentation in file systems. This is obtained by reorganizing the contents of the disk to store the pieces of each file close together and contiguously while creating larger regions of free space. Here we use a similar idea to (coherently) compress quantum information in the quantum memory M during its transfer from V. This results in a more efficient storing of messages, which saves valuable memory space for the subsequent data-processing transformations.

Referring to Refs. [7,8] for details, we can summarize the scheme of control by relaxation by saying that it consists of a *downloading* stage in which C is iteratively coupled to a fixed, finite-dimensional subspace (say a qubit) M_1 of M that is reprepared into a fiduciary state $|0\rangle_{M_1}$ after each iteration. The ℓth step of this process is described by a unitary downloading operation W_ℓ, which for large ℓ moves arbitrary states $|\psi\rangle_{C\overline{C}}$ of the system into the memory, that is,

$$W_\ell|\psi\rangle_{C\overline{C}} \otimes |0\rangle_M \approx |0\rangle_{C\overline{C}} \otimes |\Phi(\psi)\rangle_M, \tag{1}$$

with $|\Phi(\psi)\rangle_M$ being a linear function of the input state $|\psi\rangle_{C\overline{C}}$. They are then controlled in M and moved back to the system in an *uploading stage* that reverses the process (1). It is worth stressing that the transformations W_ℓ are known and are independent from the input state of the system.

This introduction seems to suggest that indeed $M = C\overline{C}$ is large enough to contain images of all possible states. This is not the case, however, as states are only transferred asymptotically and for intermediate ℓ the downloading operator W_ℓ is generating entanglement between $C\overline{C}$ and M. However, by introducing an orthonormal basis $\{|k\rangle_{C\overline{C}}\}$ of $C\overline{C}$, a generic state $|\psi\rangle = \sum_k \alpha_k |k\rangle_{C\overline{C}}$ after ℓ steps can be written as

$$W_\ell \sum_k \alpha_k |k\rangle_{C\overline{C}} \otimes |0\rangle_M = \sum_{kk'} \alpha_k \omega_{kk'}^{(\ell)} |k'\rangle_{C\overline{C}} \otimes \left|\xi_{kk'}^{(\ell)}\right\rangle_M, \tag{2}$$

with $|\xi_{kk'}^{(\ell)}\rangle_M$ being a set of $(\dim C\overline{C})^2$ not-necessarily orthogonal vectors of M. Independently of the value of ℓ, the states $\{|\xi_{kk'}^{(\ell)}\rangle_M\}_{kk'}$ span a space of dimension smaller than or equal to $(\dim C\overline{C})^2$: They can thus be fitted into a subsystem M_0 of M which is twice as large as $C\overline{C}$. Therefore, by including an extra *defragmentation* step into the protocol of Ref. [7], the memory can be kept at a finite size. Explicitely, we can write $M = M_0 \otimes M_1$. Then the defragmentation consists of operating on the memory with a unitary transformation which maps the $|\xi_{kk'}^{(\ell)}\rangle_M$ into states of the form $|\tilde{\xi}_{kk'}^{(\ell)}\rangle_{M_0} \otimes |0\rangle_{M_1}$ with

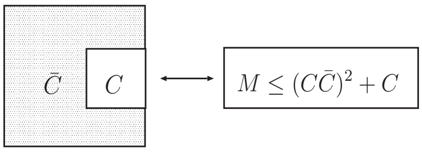

FIG. 1. Schematics of the control by relaxation scheme. The control on the large system $V = C \cup \overline{C}$ is exerted through an auxiliary (fully controllable) quantum memory M, which is directly coupled to the subsystem C.

$|0\rangle_{M_1}$ being the fiduciary state of the downloading stage, while the $|\tilde{\xi}^{(\ell)}_{kk'}\rangle_{M_0}$ are instead characterized by having the same mutual scalar product as the $|\xi^{(\ell)}_{kk'}\rangle_M$, that is,

$$ {}_{M_0}\langle\tilde{\xi}^{(\ell)}_{k''k'''}|\tilde{\xi}^{(\ell)}_{kk'}\rangle_{M_0} = {}_{M}\langle\xi^{(\ell)}_{k''k'''}|\xi^{(\ell)}_{kk'}\rangle_{M}, \quad (3) $$

for all k, k', k'', and k'''. The whole procedure can be iterated easily by observing that at the $(\ell+1)$th step the state of the system can be still described as in Eq. (2) for a proper choice of the vectors $|\xi^{(\ell+1)}_{kk'}\rangle_M$.

It is worth noticing that the defragmentation procedure presented here also finds useful application in the context of spin network communication [13]. Indeed generalizing the result of the end-gate protocol of Refs. [5,14] to the multiexcitation sector case shows that the memory-assisted transmission scheme of Ref. [15] can be implemented with finite resources.

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